Clifford Residues and Charge Quantization

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Abstract

We derive the quantization of action, particle number, and *electric* charge in a Lagrangian spin bundle over $\mathbb{M} \equiv \mathbb{M}_{\#} \backslash \cup D_J$, Penrose's conformal compactification of Minkowsky space, with the world tubes of massive particles removed.

Our Lagrangian density, $\mathcal{L}_g$, is the spinor factorization of the Maurer-Cartan 4-form Ω^4; it's action, S_g, measures the covering number of the 4 *internal* $u(1) \times su(2)$ phases over external spacetime $\mathbb{M}$. Under PTC symmetry, $\mathcal{L}_g$ reduces to the second Chern form $TrK_L \wedge K_R$ for a left $\oplus$ right chirality spin bundle. We prove a *residue theorem* for $gl(2, \mathbb{C})$-valued forms, which says that, when we "sew in" singular loci D_J over which the $u(1) \times su(2)$ phases of the matter fields have some extra twists compared to the **8** vacuum modes, the additional contributions to the action, electric charge, lepton and baryon numbers are all *topologically quantized*. Because left and right chirality 2-forms are *chiral dual,* forms are quantized over their *dual* cycles. Thus it is the interaction $c_2(E)$, with a globally nontrivial *magnetic* field, that forces *electric fields* to be topologically quantized over *spatial* 2 cycles, $\int_{\mathbb{S}^2} K_{or} e^{\theta} \wedge e^{\varphi} = 4\pi N$.

1 Introduction

Yang-Mills monopoles [**?**] have topologically quantized *magnetic* charges because it is the *magnetic* parts, $K_{jk} e^j \wedge e^k$ $(j, k = 1, 2, 3)$, of their $su(2)$-valued spin-curvature 2 forms, $K = \mathbf{d}\Omega + \Omega \wedge \Omega$, that "wrap" integrally around *spatial* 2-cycles. For electric charge to be topologically quantized, the *electric* field would have to wrap integrally about its *dual (spatial)* cycle,

$$\int_{\mathcal{S}_2} K_{0r} e^{\theta} \wedge e^{\varphi} = 4\pi N;$$

Gauss' law.

The *instantons* and *dyons* of Yang Mills theories, which possess (anti-) *self-dual* curvatures, also possess nonzero electric fields, $K_{0j} = \pm\epsilon_j^{k\ell} K_{k\ell}$, and electric charges that are quantized because their magnetic charges are. However, they live in a *Euclidean* four-space. Any analogous construction in Minkowsky space, where $** = -1$, must have imaginary (anti-) self-dual curvatures $*K = \pm iK$.

We exhibit a model here in which Left and Right spin curvatures are *imaginary chiral dual*: $K_R = \pm i * K_L$. Localized chiral dual solutions are dyons with half-integral units of electric and magnetic charges. For PT *antisymmetric* (PT_A) solutions, the electric semi-charges add, while the magnetic semi-charges cancel, thus binding together the left and right chiral halves into a bispinor particle.

In the work of Van der Waerden [**?**], Sachs [**?**], Penrose [**?**], and Keller [**?**], [**?**], it becomes clear that geometric and Fermionic fields are the integral and half-integral sectors of one unified spin-4 tensor field.

In a companion paper [**?**] (see also [**?**]), we exhibited a grand-unified Lagrangian density,

$$\mathcal{L}_g = i \int_{\mathbb{M}} \mathbf{d}\varsigma^{\pm}\xi_{\mp} \wedge \chi^{\pm}\mathbf{d}\eta_{\pm} \wedge \mathbf{d}\chi^{\mp}\eta_{\pm} \wedge \varsigma^{\mp}\mathbf{d}\xi_{\pm}, \tag{1}$$

(*sum over all neutral sign combinations*) invariant under the group E_P of *passive* Einstein transformations; Sachs' [**?**] term for the global extension of the Poincaré group to a Friedmann universe. E_P transformations connect the *same* physical state in the moving frames of different observers. In the PTC-symmetric *geometrical optics* (*g.o.*) regime in $\mathbb{M} = \mathbb{M}_{\#} \setminus \cup D_J$, outside the *singular loci* D_J, $\mathcal{L}_g$ reduces to the *Maurer-Cartan* 4 *form.* This gives a natural topological action

$$S_g = i \int_{\mathbb{M}} Tr \Omega^L \wedge \Omega_L \wedge \Omega^R \wedge \Omega_R \equiv i \int_{\mathbb{M}} \widehat{\mathcal{L}}_g, \tag{2}$$

which measures the covering number of spin space over spacetime, and comes in quantized units.

$\mathcal{L}_g$ of (1) is not unique—but its action S_g (2) does have a desirable feature: The terms in S_g decompose into effective *electroweak, strong,* and *gravitational* potentials and curvatures, together with their proper field actions [**?**]. We show here, using *spin residues*—"winding numbers" of $gl\,(2, \mathbb{C})$-valued forms about each codimension J, singularity D_J—that these *actions* and charges are *topologically quantized.*

The *singular loci* D_J are where $J = 1$, 2, 3, or 4 pairs of spin rays cross, forming *caustics.* Here the $gl\,(2, \mathbb{C})$ *phases* of J chiral pairs of spinors, i.e. the local, path-dependent exponents in the geometrical optics ($g.o.$) ansatz

$$\psi(x) = e^{i2(\theta^\alpha(x)+i\varphi^\alpha(x))\sigma_\alpha}\psi(0) \equiv e^{i2\varsigma^\alpha(x)\sigma_\alpha}\psi(0), \tag{3}$$

cannot be defined. This happens when

1. D_J contains a *zero* of $\psi \equiv \xi_\pm$, $\eta_\pm$, $\varsigma^\pm$, or $\chi^\pm$;

2. ψ or $\mathbf{d}\psi$ is undefined somewhere in D_J, i.e. D_J contains a *singular point* of ψ;

3. the phases of each field in (3) are *defined,* but J pairs break away from PTC conjugacy. The transformations that create these states violate the *spin isometry condition*

$$\zeta^\pm\xi_\mp = 1 = \chi^\pm\eta_\mp. \tag{4}$$

4. J of the 4 gradients in $\mathcal{L}_g$ become *linearly* dependent in D_J, and so fail to span a 4-volume element. The remaining pairs span the $(4-J)$-surface over which the J broken out fields are quantized, as we show below.

We call the row spinors $\varsigma^\mp$ and $\chi^\mp$ in (1) the *Baryonic spinors*. They must be treated as *independent variables* from the *leptonic* (column) spinors $\xi_\pm$ and $\eta_\pm$ in the variation of $\mathcal{L}_g$ within each singular domain D_J. In the companion paper [**?**], we identify codimension $J = 1$, 2, 3, and 4 *topological defects* in the multi-spinor fields with leptons, bosons, hadrons and their reaction vertices, respectively. Inside the D_1, $\mathcal{L}_g$ gives Dirac equations coupling each chiral pair of matter fields through nonlinear scatterings with the vacuum fields, thus creating the effective masses of bispinor particles [**?**].

However, it is not necessary to unravel the detailed structure of these core regions to prove that they carry *integral charges—electric charge, lepton number,* and *baryon number*—and of *action*, provided that the "inner" solutions for $\mathcal{L}_g$ match the "outer" ($g.o.$) solutions for $\widehat{\mathcal{L}}_g$ outside the singular domains, i.e. in $\mathbb{M} \equiv \mathbb{M}_\# \backslash \cup D_J$.

Below we prove a $(3+1)$-dimensional *Clifford residue theorem* for Lie-algebra-valued forms, that says each singular domain contributes integral units of action and charge for *any* Lagrangian density that is a natural 4 form. The argument breaks down into four steps:

1. Separate the action into outer (field) and inner (matter) contributions,

$$S_g = \int_{\mathbb{M}} \widehat{\mathcal{L}}_g + \int_{\cup D_J} \mathcal{L}_g = S_F + S_M.$$

2. Show that the field action for the vacuum spin bundle $\hat{\Psi}$ over the compact base space, $\mathbb{M}_{\#} \equiv \mathbb{S}^1 \times \mathbb{S}^3$, is topologically quantized.

3. Act on $\hat{\Psi}$ with topologically nontrivial *active local* Einstein (E_A) transformations that may become singular in codimension-J domains D_J.

4. Show that the resulting *field actions* and *charges* are all topologically quantized over $\mathbb{M}$.

2 Spin Connections and Maurer-Cartan Forms

We briefly review how spinors factor the "internal" Lie-algebra $gl\,(2,\mathbb{C})$ of conformal spinors (see Appendix). The affine *spin connection* Ω gives the spin-space increment that corresponds to each space-time increment, and *vice versa.* Ω is a $gl\,(2,\mathbb{C})$-valued 1 form that enters into the covariant derivative to assure covariance under coupled internal/external spin transformations in any moving frame.

We specialize below to spacetime and spin frames adapted to a *Friedmann universe;* an expanding "3 brane" $S_3\,(T)$ that, at *"cosmic" time* T, is approximately a hypersphere $\mathbb{S}^3\,(a) \subset \mathbb{R}^4$, with radius

$$a\,(T) = e^{Ta_{\#}} a_{\#} \equiv \gamma a_{\#}. \tag{5}$$

Here $a_{\#}$ is the equilibrium radius [?]; γ is the conformal *scale factor.*

The *real* radial coordinate T is not directly visible to us as observers embedded in $S_3\,(T)$. In relativistic kinematics, T is replaced by *arctime* $x^0 \subset \mathbb{S}^1$: the arclength travelled on $\widetilde{\mathbb{S}}^3$ by a photon, projected down to $\mathbb{S}^3\,(a_{\#})$, the fiducial three-sphere of stationery radius $a_{\#}$.

Arctime x^0 enters [?] as the real part of a *complex* time coordinate $z^0 \equiv x^0 + iy^0$; cosmic time $T \equiv y^0$ is the imaginary part. We do our local physics in a dilation-invariant way by projecting down to $\mathbb{M}_{\#} \equiv \mathbb{S}^1 \times \mathbb{S}^3\,(a_{\#})$, Penrose's [?] conformal compactification of Minkowsky space, with canonical (Lie-algebra) "coordinates" $x = \left(x^0, x^1, x^2, x^3\right)$.

$\mathbb{M}_{\#}$ is a very nice space on which to work, because it is a Lie group:

$$\mathbb{M}_{\#} \equiv \mathbb{S}^1 \times \mathbb{S}^3 \sim U\,(1) \times SU\,(2)\,.$$

$\mathbb{S}^3$ has two natural representations of translation, Left (L) and Right (R), that derive from Left or Right translation in $SU\,(2)$. These are the two *chiralities.*

Adding a $u\,(1)$ generator σ_0 to each, we obtain $\sigma_\alpha \in u\,(1) \times su\,(2)_L$ and $\bar{\sigma}_\alpha \in u\,(1) \times su\,(2)_R$, the *left* and *right* Lie algebras. These must be viewed as *independent generators* of chiral $U\,(1) \times SU\,(2)$. However, note that $\bar{\sigma}_\alpha$ is the *dual* Lie algebra to $\sigma_\alpha \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, under the *Clifford-Killing form* for the Minkowsky metric, $\eta_{\alpha\beta} \equiv diag\,(1, -1, -1, -1)$:

$$\{\sigma_\alpha, \bar{\sigma}_\beta\} \equiv \sigma_\alpha \bar{\sigma}_\beta + \sigma_\beta \bar{\sigma}_\alpha = 2\eta_{\alpha\beta}\sigma_0. \tag{6}$$

$$\sigma^\alpha = \bar{\sigma}_\alpha; \quad \sigma^\rho \sigma_\rho = -2, \tag{7}$$

is the Lorenz-invariant form.

We may thus define the *Clifford product* of "spinorized" tangent vectors $a, b \in T\mathbb{M}_\#$,

(8)

$$a = a^\alpha \sigma_\alpha,$$
$$\bar{b} = b^\beta \bar{\sigma}_\beta : 12\left(a\bar{b} + b\bar{a}\right) = \eta_{\alpha\beta} a^\alpha b^\beta \sigma_0 \equiv a_\beta b^\beta \sigma_{0.}$$

This is the *scalar* σ_0 in the Lie algebra times the *Minkowsky* product of the vectors. Note that the Clifford scalar is picked *out* by the *Trace:*

$$12Tr\left(a\bar{b}\right) = a_\beta b^\beta = a_0 b^0 - a_1 b^1 - a_2 b^2 - a_3 b^3. \tag{9}$$

In curved spacetime (A11), the $\eta_{\alpha\beta}$ are replaced by the metric coefficients $g_{\alpha\beta}$.

The columns of spin frames (A5) are a basis for the fundamental L and R chirality spinors $\xi_\pm(x)$ and $\eta_\pm(x)$ painted on $\mathbb{M}_\#$ by the *spinorization maps*

(10)

$$S : g_\pm(x) \equiv \exp\left(i2a_\# x^\alpha \sigma_\alpha^\pm\right) : \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow U(1)_\pm \times SU(2)_L$$
$$\bar{S} : \bar{g}_\pm(x) \equiv \exp\left(i2a_\# x^\alpha \bar{\sigma}_\alpha^\pm\right) : \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow U(1)_\pm \times SU(2)_R\,,$$

where $\sigma_\alpha^\pm \equiv (\pm\sigma_0, \sigma)$. Their infinitesimal versions are the L- and R-invariant *Maurer-Cartan* 1 *forms*:

(11)

$$TS(x) \equiv g_\pm^{-1} dg_\pm(x) = i2a_\# \sigma_\alpha^\pm e^\alpha(x) : e_\beta(x) \longrightarrow i2a_\# \sigma_\beta^\pm(x)$$
$$T\bar{S}(x) \equiv \bar{g}_\pm^{-1} d\bar{g}_\pm(x) = i2a_\# \bar{\sigma}_\alpha^\pm \bar{e}^\alpha(x) : \bar{e}_\beta(x) \longrightarrow i2a_\# \bar{\sigma}_\beta^\pm(x)\,.$$

The Maurer-Cartan 1 forms give the images in the "internal" Lie algebras $u(1)_\pm \times su(2)_L$ and $u(1)_\pm \times su(2)_R$ of infinitesimal L and R translations on $\mathbb{M}_\#$; i.e. the canonical spin-space increments that accompany a spacetime translation on $\mathbb{M}_\#$.

In the presence of a *source,* a translation is accompanied by *active local* spin space increments $\ell(x)$ and $r(x)$ in the reference frame of an observer O. O then experiences the *vector potentials*

(12)

$$\begin{aligned}\Omega_L \equiv \ell^{-1}\mathbf{d}\ell = \Omega_{L\alpha}e^{\alpha};\\ \Omega_{L\alpha} = \ell^{-1}\partial_{\alpha}\ell; \qquad \Omega_R \equiv r^{-1}\mathbf{d}r.\end{aligned}$$

The Lie-algebra-valued 1 forms, or *spin connections* Ω_L and Ω_R are the *Maurer-Cartan* 1 *forms* for local $Gl(2,\mathbb{C})$ *deformations* $\ell(x)$ and $r(x)$ of the canonical maps (2) of spacetime into spin space (see (A10) below). *Regular g.o.* perturbations do not change the rank of the mapping ψ of physical space to spin space.

3 Vector Potentials from Active Local Spin Transformations

Active local (E_A) transformations represent both local dilation/boost flows and local $U(1)\times SU(2)$ phase flows in the *geometrical optics* (*g.o.*) regime. E_A transformations on the tetrads (A8), (A9) are presented as *complexified* chiral

$$U(1)\times SU(2) \overset{\mathbb{C}}{\Longrightarrow} GL(2,\mathbb{C})$$

spin transformations on the canonical spin frames:

(13)

$$\begin{aligned}\ell(z) = \ell(0)\,L(z) \equiv \ell(0)\exp i2\left(\theta_L^{\alpha}(z) + i\varphi_L^{\alpha}(z)\right)\sigma_{\alpha} \equiv \ell(0)\,e^{i2\varsigma_L^{\alpha}(z)\sigma_{\alpha}}\\ \bar{r}(z) = \bar{R}(z)\,\bar{r}(0) \equiv \exp\left(i2\left(\theta_R^{\alpha}(z) + i\varphi_R^{\alpha}(z)\right)\sigma_{\alpha}\right)\bar{r}(0) \equiv e^{i2\varsigma_R^{\alpha}(z)\sigma_{\alpha}}\bar{r}(0),\end{aligned}$$

where we may take $\ell(0) = \sigma_0 = r(0)$.

In a spin bundle E with a momentum flow $y_{\beta}(x)$, the Cartan moving spin frames (3) are *path dependent* functions of x. The $\theta_L^{\alpha}(x)$ are the coefficients of the *anti-Hermitian* (aH) matrices $i2\sigma_{\alpha}$ that generate (local) *unitary* $U(1)\times SU(2)_L$ spin transformations. Their differentials are the *electroweak vector potentials:*

$$i2\mathbf{d}\theta^{\alpha}\sigma_{\alpha} \equiv W_{\beta}e^{\beta}.$$